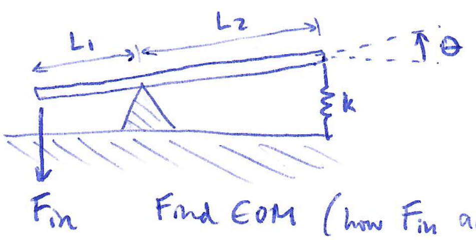
ME 4555 - Lecture 4 - Transformers

Many mechanical systems involve multiple inertias (translating or rotating) that are constrained to move together. This introduces <u>kinematic constraints</u>.

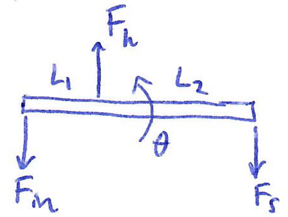
First, an important point. Newton's Second law holds in any <u>inertial</u> frame (i.e. non-accelerating). This means our <u>reference point</u> (point about which we measure torques and accelerations) shouldn't accelerate or rotate, with <u>the exception of the center of mass</u>.

<u>Example</u>: Lever:



Find EOM (how F_{in} affects θ)

<u>FBD</u>:



Using small-angle approximation, we have $\boxed{F_s = k L_2 \theta.}$

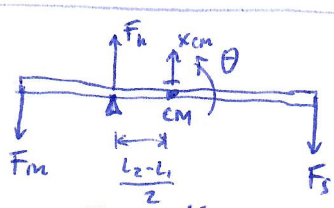
assume moment of inertia about hinge is J_h.

<u>Simplest approach</u>: torque balance about <u>hinge</u>. Hinge is not accelerating (inertial) therefore $J_h \ddot{\theta} = \sum T_{ext.} \Rightarrow \boxed{J_h \ddot{\theta} = F_{in} L_1 - F_s L_2}$

<u>Note</u>: mass m and force F_h do not appear at all. We can substitute the small-angle formula above for F_s to obtain $J_h \ddot{\theta} + k L_2 \theta = F_{in} L$.

<u>Using center of mass instead</u>

<u>FBD</u>



$$\underbrace{\left(\frac{L_1+L_2}{2}\right) F_{in} - \left(\frac{L_2-L_1}{2}\right) F_h - \left(\frac{L_1+L_2}{2}\right) F_s}_{\text{torques about CM}} = \underbrace{J_{CM}}_{\text{moment of inertia about CM.}} \cdot \ddot{\theta} \qquad (1)$$

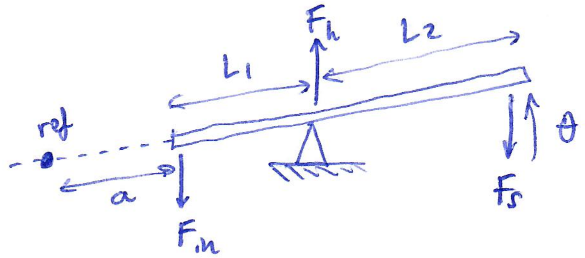
Force balance: $F_h - F_{in} - F_s = m \ddot{x}_{CM}$. But $x_{cm} = \left(\frac{L_2 - L_1}{2}\right)\theta$.

$\Rightarrow F_h = F_{in} + F_s + m\left(\frac{L_2-L_1}{2}\right)\ddot{\theta}$. Substituting into (1) (to eliminate F_h) and simplifying:

$$\boxed{\left(J_{cm} + m\left(\frac{L_2-L_1}{2}\right)^2\right)\ddot{\theta} = F_{in} L_1 - F_s L_2}$$

By the parallel axis theorem, we have $J_h = J_{cm} + m\left(\frac{L_2-L_1}{2}\right)^2$ so the solution ...

[Bonus]. it's also possible to use a point *other* than the hinge or the center of mass, but things get complicated. Let's pick a point a distance "a" to the right of the beam, but that rotates with it:



Moment of inertia about ref is (parallel axis thm)

$$J_r = J_{cm} + m\left(a + \frac{L_1+L_2}{2}\right)^2.$$

the torque balance looks like: (about ref. point)

$$\underbrace{F_h(a+L_1) - F_{in}\,a - F_s(a+L_1+L_2)}_{\text{external torque about ref.}} = \underbrace{J_r}_{\substack{\text{moment of inertia}\\ \text{about reference}}}\ddot{\theta} + \underbrace{m\left(a+\frac{L_1+L_2}{2}\right)\underbrace{\left[-(a+L_1)\ddot{\theta}\right]}_{\substack{\text{vertical acceleration}\\ \text{of reference pt.}}}}_{\text{"fictitious" torque about C.M.}}$$

the extra term on the right is due to the fact that our reference point is accelerating. If we simplify this expression, all the "a"'s cancel, and we are left with:

$$\left(J_{cm} + m\left(\frac{L_2-L_1}{2}\right)^2\right)\ddot{\theta} = F_{in}L_1 - F_s L_2$$

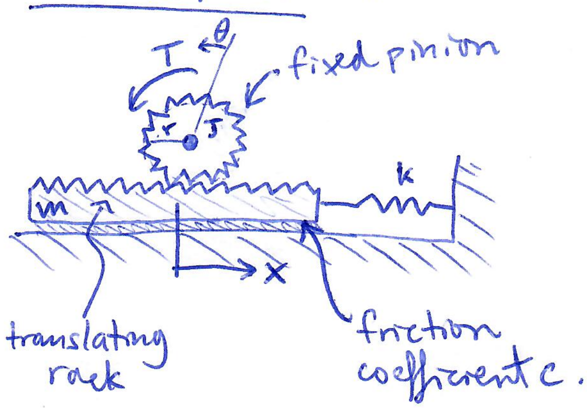
(same as before!)

Note that the extra fictitious torque term $-m(a+L_1)\left(a+\frac{L_1+L_2}{2}\right)\ddot{\theta}$ is zero precisely when we pick the reference point to be

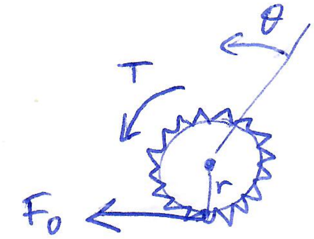
$a = -L_1$ (reference @ hinge) or $a = -\frac{L_1+L_2}{2}$ (reference @ CM).

In general, our small-angle assumption assumes only vertical motion, so our correction term is just to account for the acceleration (translational) of our reference. If we used a rotating frame of reference (say, fixed to the beam without small angles), we would also have to include extra terms for Euler, Centrifugal, and Coriolis fictitious forces.

Rack & pinion:

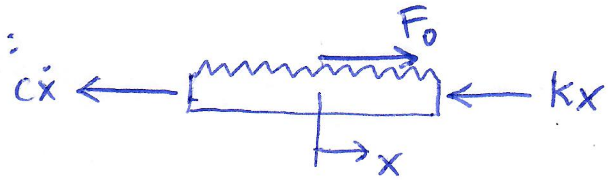


Pinion:



$$J\ddot{\theta} = T - rF_0 \quad \text{(torque balance)}$$

Rack:



$$m\ddot{x} + c\dot{x} + kx = F_0 \quad \text{(force balance)}$$

$$\boxed{\begin{aligned} J\ddot{\theta} &= T - rF_0 \\ m\ddot{x} + c\dot{x} + kx &= F_0 \end{aligned}}$$

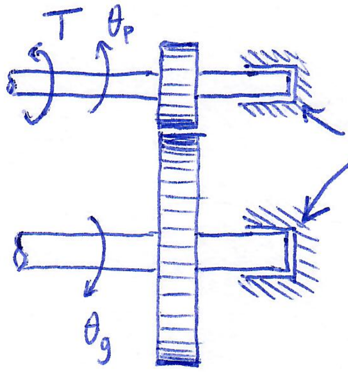
This is two equations, but we have three unknowns (θ, x, F_0) so what is missing? the rack and pinion move together!

If the rack moves x, the pinion must rotate an angle θ such that $\boxed{r\theta = x}$ (arc-length = distance) because there is no slip.

We can use this information to eliminate a variable. For example, we may be interested in how T relates to x. then use $\ddot{\theta} = \frac{1}{r}\ddot{x}$ to eliminate $\ddot{\theta}$ and also eliminate F_0.

$$\Rightarrow \left\{ \begin{aligned} \frac{1}{r} J \ddot{x} &= T - rF_0 \\ m\ddot{x} + c\dot{x} + kx &= F_0 \end{aligned} \right\} \Longrightarrow \boxed{\left(m + \frac{J}{r^2}\right)\ddot{x} + c\dot{x} + kx = \frac{T}{r}}$$

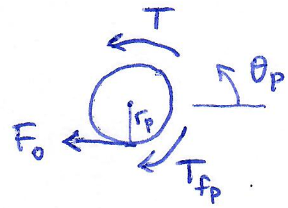
Gear + pinion (side view)



how is T related to θ_g?

FBD (view from left)

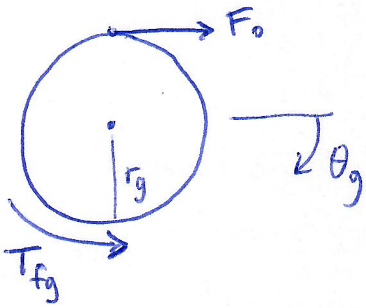
pinion:



$$J_p \ddot{\theta}_p = T - F_0 r_p - \underbrace{T_{fp}}_{b\dot{\theta}_p \text{ (friction)}}$$

$$\Rightarrow \boxed{J_p \ddot{\theta}_p + b\dot{\theta}_p = T - F_0 r_p} \qquad (1)$$

gear:



$$J_g \ddot{\theta}_g = F_0 r_g - \underbrace{T_{fg}}_{b\dot{\theta}_g \text{ (friction)}}$$

$$\Rightarrow \boxed{J_g \ddot{\theta}_g + b\dot{\theta}_g = F_0 r_g} \qquad (2)$$

Kinematic constraint: arc-lengths are equal: $\boxed{r_p \theta_p = r_g \theta_g}$
(no slipping)

substitute $\theta_p = \frac{r_g}{r_p}\theta_g$ into (1)

and multiply (2) by $\frac{r_p}{r_g}$ to

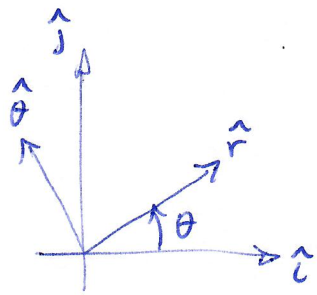
obtain:

$$\left\{\begin{aligned} &\frac{r_g}{r_p} J_p \ddot{\theta}_g + \frac{r_g}{r_p} b \dot{\theta}_g = T - F_0 r_p \\ &\frac{r_p}{r_g} J_g \ddot{\theta}_g + \frac{r_p}{r_g} b \dot{\theta}_g = F_0 r_p \end{aligned}\right\}$$

sum to cancel $F_0 r_p$ and obtain:

$$\boxed{\underbrace{\left(\frac{r_g}{r_p} J_p + \frac{r_p}{r_g} J_g\right)}_{\text{"equivalent" inertia } J_{eq}} \ddot{\theta}_g + b \underbrace{\left(\frac{r_g}{r_p} + \frac{r_p}{r_g}\right)}_{\text{"equivalent" friction } b_{eq}} \dot{\theta}_g = T}$$

Bonus: EOM in a rotating frame of reference.



$(\hat{i}, \hat{j})$ = unit vectors in inertial (fixed) frame

$(\hat{r}, \hat{\theta})$ = unit vectors in rotating frame (fixed to object)

Consider a particle of mass m a distance ρ in the $\hat{r}$ direction.

$$\vec{x} = \rho \hat{r} = \rho(\cos\theta\, \hat{i} + \sin\theta\, \hat{j})$$

Note: $\hat{r} = \cos\theta\, \hat{i} + \sin\theta\, \hat{j}$ and $\hat{\theta} = -\sin\theta\, \hat{i} + \cos\theta\, \hat{j}$.

$$\dot{\hat{r}} = -\sin\theta\, \dot{\theta}\, \hat{i} + \cos\theta\, \dot{\theta}\, \hat{j} = \underbrace{(-\sin\theta\, \hat{i} + \cos\theta\, \hat{j})}_{\hat{\theta}}\, \dot{\theta}$$

$$\dot{\hat{\theta}} = -\cos\theta\, \dot{\theta}\, \hat{i} - \sin\theta\, \dot{\theta}\, \hat{j} = -\hat{r}\, \dot{\theta}$$

$$\underset{\text{velocity}}{\dot{\vec{x}}} = \underbrace{\dot{\rho}\hat{r} + \rho \dot{\hat{r}}}_{\text{chain rule}} = \underset{\text{radial speed}}{\dot{\rho}}\, \hat{r} + \underbrace{\rho\dot{\theta}}_{\text{angular speed}}\, \hat{\theta}$$

$$\underset{\text{acceleration}}{\ddot{\vec{x}}} = \frac{d}{dt}(\dot{\rho}\hat{r} + \rho\dot{\theta}\hat{\theta}) = \underbrace{\ddot{\rho}\hat{r} + \dot{\rho}\dot{\theta}\hat{\theta}}_{\text{chain rule}} + \underbrace{\dot{\rho}\dot{\theta}\hat{\theta} + \rho\ddot{\theta}\hat{\theta} - \rho\dot{\theta}^2\hat{r}}_{\text{chain rule.}}$$

$$\Rightarrow \ddot{\vec{x}} = \underbrace{(\ddot{\rho} - \rho\dot{\theta}^2)}_{\text{in radial direction}}\hat{r} + \underbrace{(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})}_{\text{in perpendicular direction}}\hat{\theta}$$

$\ddot{\rho}$: radial acceleration due to ρ changing

$-\rho\dot{\theta}^2$: centrifugal force due to rotation

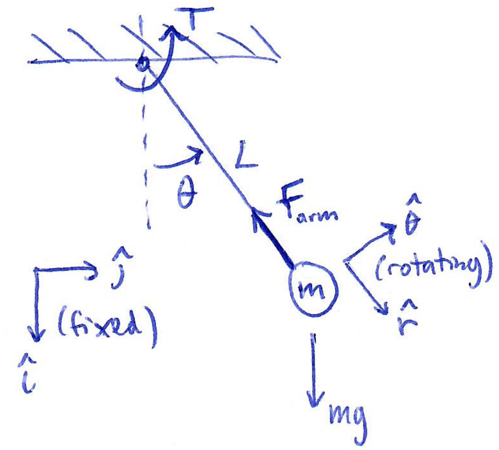
$2\dot{\rho}\dot{\theta}$: Coriolis force due to ρ + θ changing

$\rho\ddot{\theta}$: Euler force due to θ accelerating

in radial direction

in perpendicular direction

Simple case



Using Newton's 2nd law with $\dot\rho = 0$, $\ddot\rho = 0$ since rod is rigid, we have:

$$m\ddot{\vec{x}} = \underbrace{-mL\dot\theta^2\hat r + mL\ddot\theta\hat\theta}_{\text{mass x acceleration}} = mg\underset{(\cos\theta\,\hat r - \sin\theta\,\hat\theta)}{\hat i} + \frac{T}{L}\hat\theta - F_{arm}\hat r$$

in $\hat r$ direction: $\quad -mL\dot\theta^2 = mg\cos\theta - F_{arm}$

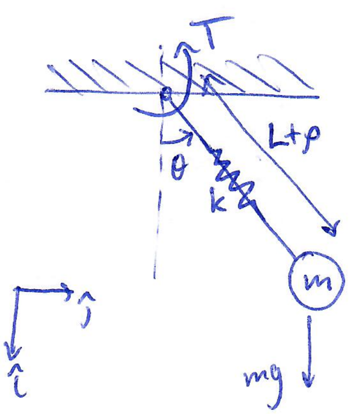
in $\hat\theta$ direction: $\quad mL\ddot\theta = -mg\sin\theta + \frac{T}{L}$

EOM $\Rightarrow$

$$\begin{cases} \ddot\theta + \frac{g}{L}\sin\theta = \frac{T}{mL^2} & \leftarrow \text{same eqn we found last time.} \\ F_{arm} = mL\dot\theta^2 + mg\cos\theta & \leftarrow \text{eqn for tension force in the pendulum arm.} \end{cases}$$

Complicated case

: Pendulum with spring. Spring only extends/shrinks the arm, it does not bend.



($\rho = 0$ corresponds to zero spring force)

$$m\ddot{\vec{x}} = m(\ddot\rho - (L+\rho)\dot\theta^2)\hat r + m(2\dot\rho\dot\theta + (L+\rho)\ddot\theta)\hat\theta$$

$$\Sigma\,\text{forces} = \underbrace{mg(\cos\theta\,\hat r - \sin\theta\,\hat\theta)}_{\text{gravity}} + \underbrace{\frac{T}{L+\rho}\hat\theta}_{\text{Torque}} - \underbrace{k\rho\,\hat r}_{\text{spring}}$$

EOM (messy!):

$$\begin{cases} m\ddot\rho - m(L+\rho)\dot\theta^2 - mg\cos\theta + k\rho = 0 & (\hat r \text{ direction}) \\ m(L+\rho)\ddot\theta + 2m\dot\rho\dot\theta + mg\sin\theta = \frac{T}{L+\rho} & (\hat\theta \text{ direction}) \end{cases}$$